CHAPTER

Decomposition Theorems In Hilbert Spaces

Annihilator OR Orthogonal Complement#

Let A be any subset of a Hilbert space H. The set of all vectors which are orthogonal to A is called the annihilator of A or orthogonal Complement of A and is denoted by $A^{\perp}$.

Thus

$$A^{\perp} = \{x \in H : x \perp A\}$$

$$= \{x \in H : \langle x, y\rangle = 0 \quad \forall y \in A\}$$

Consequences of Definition#

following results are clear from the definition:

1) : $\{0\}^{\perp} = H$ $\quad H^{\perp} = \{0\}$

2) $A \cap A^{\perp} \subseteq \{0\}$

3) $A \subseteq B \Rightarrow A^{\perp} \supseteq B^{\perp}$

4) $A^{\perp}$ is a closed linear subspace of H

Theorem# Let A & B be subsets of Hilbert space H. Then.

(i) $A \subseteq A^{\perp\perp}$

(ii) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$

(iii) $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$ & $A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$

(iv) $A^{\perp} = A^{\perp\perp\perp}$

(V) $A \cap A^{\perp} \subseteq \{0\}$

(VI) $A^{\perp}$ is a closed subspace of H.

Proof (i) Let $x \in A$. Then $\langle x, y\rangle = 0 \;\; \forall y \in A^{\perp}$

Hence $x \in A^{\perp\perp}$

$\Rightarrow A \subseteq A^{\perp\perp}$

(ii) Let $A \subseteq B$ and $x \in B^{\perp}$

Then. $\langle x, y\rangle = 0 \quad \forall y \in B$

Since $A \subseteq B$

Therefore

$\langle x, y\rangle = 0 \quad \forall y \in A$

$\Rightarrow x \in A^{\perp}$ i.e $B^{\perp} \subseteq A^{\perp}$

(iii) Since

$A \subseteq A \cup B \quad , B \subseteq A \cup B$

from (ii)

$(A \cup B)^{\perp} \subseteq A^{\perp}$ & $(A \cup B)^{\perp} \subseteq B^{\perp}$

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp} \cap B^{\perp}$

Conversely let $x \in A^{\perp} \cap B^{\perp}$

$x \in A^{\perp}$ & $x \in B^{\perp}$ i.e

$\langle x \;\; u\rangle = 0 \;\; \forall u \in A$

$\langle x, u\rangle = 0 \;\; \forall u \in B$

Hence $\langle x \;\; y\rangle = 0 \quad \forall y \in A \cup B$

$\Rightarrow \quad x \in (A \cup B)^{\perp}$

$\Rightarrow \quad A^{\perp} \cap B^{\perp} \subseteq (A \cup B)^{\perp}$

Consequently

$(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

Next $\quad A \cap B \subseteq A \qquad A \cap B \subseteq B$

$\Rightarrow A^{\perp} \subseteq (A \cap B)^{\perp} \qquad B^{\perp} \subseteq (A \cap B)^{\perp}$

$A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$

(iv) By (i)

$$A \subseteq A^{\perp\perp}$$

By (ii)

$$A^{\perp\perp\perp} \subseteq A^{\perp}$$

Also by (i)

$$A^{\perp} \subseteq (A^{\perp})^{\perp\perp} = A^{\perp\perp\perp}$$

Hence

$$A^{\perp} = A^{\perp\perp\perp}$$

If $A \cap A^{\perp} = \phi$, then $A \cap A^{\perp} \subseteq \{0\}$

so

let $x \in A \cap A^{\perp}$

Then $x \in A$ and $x \in A^{\perp}$

So $x \perp x$

i.e $\|x\|^2 = \langle x, x \rangle = 0$

$\Rightarrow x = 0$ Hence $A \cap A^{\perp} = \{0\}$

(vi) Let $y, z \in A^{\perp}$, $a, b \in F$

Then for any $x \in A$

$\langle y, x \rangle = 0$ $\langle z\ x \rangle = 0$

$$\langle ay + bz\ x \rangle = a\langle y, x \rangle + b\langle z\ x \rangle$$
$$= 0$$

Hence $ay + bz \in A^{\perp}$. So $A^{\perp}$ is a subspace of H

Next let y be a limit point of $A^{\perp}$. Then there is a sequence $\{y_n\}$ in $A^{\perp}$ such that

$$\lim_{n \to \infty} y_n = y$$

Now $\langle y_n\ x \rangle = 0$ $\forall x \in A$

Hence $0 = \lim_{n \to \infty} \langle y_n\ x \rangle$

$$= \langle \lim_{n \to \infty} y_n\ \ x \rangle = \langle y, x \rangle$$

Hence $y \in A^{\perp}$ so $A^{\perp}$ is a closed subspace

Theorem # (Minimizing Vector)

Let A be a complete convex subset of an inner product space V and $x \in V \setminus A$. Then there is a unique $y \in A$ such that

$$\|x-y\| = \inf_{y' \in A} \|x-y'\|$$

i.e there is a unique $y \in A$ which is closest to x

Proof # Let

$$d = \inf_{y' \in A} \|x-y'\|$$

Then by definition of infimum, there is a sequence $\{y_n\}$ in A such that

$$d = \lim_{n \to \infty} \|x - y_n\|$$

We show that $\{y_n\}$ is a cauchy sequence in A. By the parallelogram law

$$\|x'-y'\|^2 = 2\|x'\|^2 + 2\|y'\|^2 - \|x'+y'\|^2$$

By replacing x' by $y_m - x$ and y' by $y_n - x$.

$$\|y_m - y_n\|^2 = 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|y_m + y_n - 2x\|^2$$

$$= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4\|\tfrac{1}{2}(y_m + y_n) - x\|^2 \rightarrow \text{Ⓐ}$$

Since A is convex, $\frac{1}{2}(y_m + y_n) \in A$ so that we have from Ⓐ

$$\|y_m - y_n\|^2 \leq 2\|y_m - y_n\|^2 + 2\|y_n - x\|^2 - 4d^2$$

$$\rightarrow 0 \quad \text{as } m, n \rightarrow \infty$$

because $\|y_m - x\| \to d$, $\|y_n - x\| \to d$.
Hence $\{y_n\}$ is a Cauchy sequence in A. Since A is complete, $y_n \to y \in A$. So

$$d = \lim_{n\to\infty} \|x - y_n\|$$
$$= \|x - \lim_{n\to\infty} y_n\|$$
$$= \|x - y\|$$

with y in A.

Uniqueness of y #

Suppose that there is another y_0 in A such that

$$d = \|x - y_0\|$$

Then again using the parallelogram law and replacing x' by $y-x$ & y' by $y_0 - x$, we have

$$\|y - y_0\| = 2\|y-x\|^2 + 2\|y_0 - x\|^2 - \|y + y_0 - 2x\|$$
$$\leq 2\|y-x\|^2 + 2\|y_0 - x\|^2 - 4\|\tfrac{1}{2}(y+y_0) - x\|^2$$

Since A is convex and $\frac{1}{2}(y+y_0) \in A$, we have

$$\|y - y_0\|^2 \leq 4d^2 - 4d^2 = 0$$

But $\|y - y_0\|^2 \geq 0$

Hence $y - y_0 = 0 \Rightarrow y = y_0$

This proves the uniqueness of y

Corollary # (Every closed subspace of). Let A be a closed subspace of a Hilbert space H and $x \in H \setminus A$. Then there is a unique $y \in A$ such that

$$\|x - y\| = \inf_{y' \in A} \|x - y'\|$$

Proof# Let A be a ~~convex~~ closed subspace of Hilbert space. Every closed subspace of a complete metric space is complete and convex. Since H is complete, A is complete and the statement ultimately coincides with the statement of Theorem and proof is similar.

Corollary# Let A be convex and complete subset of an inner product space. Then A contains a unique vector of the smallest norm

Proof# Take $x=0$ then

$$\|x-y\| = \operatorname*{Inf}_{y'\in A} \|x-y'\|$$

becomes

$$\|y\| = \operatorname{Inf}\|y'\|,\ y'\in A$$

Since A is complete, $y\in A$ and is unique, as required.

Theorem# Let A be a complete subspace of an inner product space V. Then there is a non-zero vector z in V such that

$$z\perp A$$

Proof Let x be a vector not in A i.e $x\in V/A$. Then by above theorem there is a unique vector $y\in A$ such that

$$\|x-y\| = \operatorname*{Inf}_{y'\in A}\|x-y'\|$$

Let $z = x-y$. we show that $z\perp A$.
Let y_1 be any arbitrary element of A.
We have to prove that

$$\langle z\ y_1\rangle = 0$$

Without any loss of generality we can suppose that $\|y_1\| = 1$ for otherwise we can replace y_1 by $y_1/\|y_1\|$. Now for any scalar α, we have

$$\|z - \alpha y_1\|^2 = \langle z-\alpha y_1, z-\alpha y_1\rangle$$

$$= \|z\|^2 - \alpha\langle y_1, z\rangle - \bar{\alpha}\langle z, y_1\rangle + \alpha\bar{\alpha}$$

$$= \|z\|^2 - \alpha\langle y_1, z\rangle - \bar{\alpha}\langle z, y_1\rangle + \alpha\bar{\alpha}$$

In particular for $\alpha = \langle z, y_1\rangle$

$$\|z - \alpha y_1\|^2 = \|z\|^2 - \langle z, y_1\rangle\langle y_1, z\rangle - \overline{\langle z, y_1\rangle}\langle z, y_1\rangle + \langle z, y\rangle\overline{\langle z, y_1\rangle}$$

$$= \|z\|^2 - \langle z, y_1\rangle\overline{\langle z, y_1\rangle} - \overline{\langle z, y_1\rangle}\langle z, y_1\rangle + \langle z, y\rangle\overline{\langle z, y_1\rangle}$$

$$= \|z\|^2 - |\langle z, y_1\rangle|^2$$

Also $z - \alpha y_1 = x - y - \alpha y_1$

$$= x - (y + \alpha y_1)$$

$$= x - y_2$$

where $y_2 = y + \alpha y_1 \in A$, because A is subspace of inner product space.

Hence

$$\|z\|^2 = \|x-y\|^2 \le \|z - \alpha y_1\|^2$$

Because $\|x-y\| = \inf_{y' \in A} \|x - y'\|$

$$\le \|z\|^2 - |\langle z, y_1\rangle|^2 \le \|z\|^2$$

$$\Rightarrow \quad \|z\|^2 = \|z\|^2 - |\langle z, y_1\rangle|^2$$

Therefore $\langle z, y_1\rangle = 0$

So $z \perp y_1$. Since y_1 is an arbitrary, $z \perp A$ as Required.

Remarks# Since every closed subspace of a complete

Metric space is Complete and Hilbert space is also Complete metric space (inner product space is metric space with metric induced by the norm), therefore the statement of the above theorem can given as

If A is a proper closed linear subspace of Hilbert space H, then there exists a non-zero vector z_0 in H such that $z_0 \perp A$.

Theorem# If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M+N$ is also closed.

Proof# Let z be a limit point of $M+N$. Then there $\{z_n\} = \{x_n + y_n\}$ in $M+N$, $x_n \in M$, $y_n \in N$ such that

$\lim\limits_{n\to\infty} z_n$ $$\lim_{n\to\infty} (x_n + y_n) = z$$

Since the sequence $\{z_n\}$ converges, it is a cauchy sequence, so

$$\lim_{m,n\to\infty} \|z_n - z_m\| = 0$$

But $$\|z_n - z_m\|^2 = \|x_n + y_n - x_m - y_m\|^2$$

$$= \|x_n - x_m\|^2 + \|y_n - y_m\|^2$$

(By Pythagorean Theorem law)

$\to 0$ as $m, n \to \infty$

Hence $\|x_n - x_m\| \to 0$ $\quad \|y_n - y_m\| \to 0$

So $\{x_n\}$ is a cauchy sequence in M and $\{y_n\}$ is a cauchy sequence in N. Since M, N are closed, so there points $x \in M$, $y \in N$ s.t

$\lim\limits_{n\to\infty} x_n = x$ & $\lim\limits_{n\to\infty} y_n = y$

But then $z = \lim\limits_{n\to\infty} (x_n + y_n) = x + y \in M+N \Rightarrow M+N$ is closed

Direct Sum # A linear space V is direct sum of of its subspaces V_1 & V_2 if every element v of V is uniquely expressed as

If $V = V_1 \oplus V_2$ ~~and~~ then $v = v_1 + v_2$ $\quad v_1 \in V_1 \quad v_2 \in V_2$

$$V_1 \cap V_2 = \{0\}$$

Direct sum is denoted by $V_1 \oplus V_2$

v_1 is called an algebric complement of v_2 and vice-versa and V_1, V_2 is called a complementry pair of subspaces in V

Theorem # Let A be a complete subspace of an inner product space V. Then

$$V = A \oplus A^{\perp}$$

Proof # Since A is complete and, being a subspace, is convex (every subspace of a linear space is convex). Therefore by minimizing vector Theorem there is a unique vector y in A such that

$$\|x - y\| = \inf_{y' \in A} \|x - y'\| \qquad \text{for each } x \in V \setminus A$$

Put $z = x - y$. Then by above Theorem

$$z \perp A$$

and so $z \in A^{\perp}$ which is a subspace of V

Hence

$$x = y + z, \quad y \in A \quad z \in A^{\perp} \quad \rightarrow (1)$$

To see that that the expression (1) is unique, suppose that

$$x = y_1 + z_1$$

Then $y - y_1 = z_1 - z \in A \cap A^{\perp} = \{0\}$

So $y = y_1$ & $z_1 = z$

Hence expression (1) is unique.

Now if $x' \in A \subseteq V$, then

$$x' = x' + 0 \in A \oplus A^{\perp}$$

$\because z, z_1 \in A^{\perp}$ & $A^{\perp}$ is subspace
$\therefore z_1 - z \in A^{\perp}$
$\Rightarrow y - y_1 \in A^{\perp}$
By $y_1, y \in A$
$\Rightarrow y_1 - y \in A$
$\Rightarrow z_1 - z \in A$

and this expression is also unique.

Thus every element of V can be expressed uniquely as sum of elements of A & $A^{\perp}$.

Hence

$$V = A \oplus A^{\perp}$$

Remarks # For any complete subspace A of an inner product space V the space $A^{\perp}$ of is called the orthogonal complement of A. In particular if A is a closed subspace of Hilbert space H, then $A^{\perp}$ is the orthogonal complement of A in H.

Corollary #1 Let A be a closed subspace of Hilbert space H. Then

$$H = A \oplus A^{\perp}$$

Proof # Since A, as a closed subspace of Hilbert space H which is always complete space, is complete, therefore the corollary is proved by above theorem.

OR

It can be proved as an independent theorem.

Proof # $\because$ A is closed subspace of Hilbert space H.

$\therefore$ $A^{\perp}$ is also closed subspace of H.

$\Rightarrow$ $A + A^{\perp}$ is also a closed linear subspace of H

Moreover $A \cap A^{\perp} = \{0\}$

We show that $H = A \oplus A^{\perp}$.

obviously $A + A^{\perp} \subseteq H$

Suppose that $H \neq A + A^{\perp}$ i.e $H \not\subseteq A + A^{\perp}$

Then the $A + A^{\perp}$ is a proper subspace of H and is also closed.

Therefore by a previous theorem there is a non-zero vector $z_0 \in H$ such that

$z_0 \perp (A + A^{\perp})$

Here $z_0 \in H \setminus (A + A^{\perp})$

so $z_0 \in (A + A^{\perp})^{\perp} \longrightarrow (A)$

Now $A \subseteq A + A^{\perp}$

$\Rightarrow$ ~~$A^{\perp}$~~ $\subseteq$

$(A + A^{\perp})^{\perp} \subseteq A^{\perp} \longrightarrow (1)$

Also $A^{\perp} \subseteq A + A^{\perp}$

$(A + A^{\perp})^{\perp} \subseteq A^{\perp\perp} \longrightarrow (2)$

By (1) & (2)

$(A + A^{\perp})^{\perp} \subseteq A^{\perp} \cap A^{\perp\perp}$

Hence by (A)

$z_0 \in (A + A^{\perp})^{\perp} \subseteq A^{\perp} \cap A^{\perp\perp}$

$z_0 \in A^{\perp} \cap A^{\perp\perp} = \{0\}$

$\Rightarrow z_0 = 0$ which is impossible because z_0 is non-zero vector. Hence

$H = A^{\perp} + A$

Thus $H = A \oplus A^{\perp}$

Corollary # 2 For any complete subspace A of an inner product space V

$$A = A^{\perp\perp}$$

OR

For any closed subspace A of Hilbert space H

$$A = A^{\perp\perp}$$

In face For any ~~comp clo~~ subspace A of Hilbert space A, we have

A is closed $\Leftrightarrow A = A^{\perp\perp}$

Proof # We know that

$A \subseteq A^{\perp\perp} \longrightarrow (1)$

Conversely suppose that $x \in A^{\perp\perp}$.

Since $V = A \oplus A^{\perp}$

There are elements $y \in A \subseteq A^{\perp\perp}$, $z \in A^{\perp}$ such that

$$x = y + z$$

$$z = x - y$$

But $x - y \in A^{\perp\perp}$, because $A^{\perp\perp}$ is a subspace.

Hence

$$x - y \in A^{\perp} \cap A^{\perp\perp} = \{0\}$$

$$\Rightarrow \quad x = y \in A$$

Hence $A^{\perp\perp} \subseteq A$

Therefore $A = A^{\perp\perp}$

OR

Let $A = A^{\perp\perp}$

Since for any subspace A, $A^{\perp}$ is closed, $A^{\perp\perp}$ is also closed. Hence A is closed.

Conversely suppose that A is closed.

Let $x \in A$

$$\Rightarrow x \perp A^{\perp}$$

$$\Rightarrow x \in A^{\perp\perp}$$

$$\Rightarrow A \subseteq A^{\perp\perp} \rightarrow (1)$$

Let $z \in A^{\perp\perp}$

Since A is closed, therefore by above Theorem

$$H = A \oplus A^{\perp}.$$

So $z = x + y \quad x \in A \quad y \in A^{\perp}$

Now $x \in A \subseteq A^{\perp\perp}$

So $y = z - x \in A^{\perp\perp}$.

But $y \in A^{\perp}$ so $y \in A^{\perp} \cap A^{\perp\perp} = \{0\}$

$$\Rightarrow y = 0$$

$$\Rightarrow z = x \in A$$

Hence $A^{\perp\perp} \subseteq A \rightarrow (2)$

from (1) & (2) $\quad A = A^{\perp\perp}$

Theorem # Let A & B be closed subspace of a Hilbert space H such that $A \perp B$. Then $A+B$ is a closed subspace of H

Proof # For any subspaces A, B, $A+B$ is always subspace. We show that $A+B$ is closed.

Let z be a limit point of $A+B$

This Theorem is already proved.

* Projections In Hilbert Spaces

Let A be a closed subspace of Hilbert space H. Then we have

$$H = A \oplus A^{\perp}$$

So that each $x \in H$ is uniquely of the form

$$x = y+z \qquad y \in A,\ z \in A^{\perp}$$

Define a function $\pi : H \to A$ by

$$\pi(x) = \pi(y+z) = y$$

Then $\pi(y) = y \quad \forall\, y \in A$ i.e

$\pi/_A$ (π restricted to A) = the identity function on A

Also for each $z \in A^{\perp}$, $\pi(z) = 0$

So $A^{\perp}$ is null space π

This function is called the orthogonal projection of H onto A

Since

$$\pi^2(x) = \pi^2(y+z) = \pi(y) = y = \pi(x) \quad \forall x \in H$$

$\Rightarrow \pi^2 = \pi$

$\Rightarrow \pi$ is an idempotent function.

Every projection is Linear.

For $x_1, x_2 \in H$

$$x_1 = y_1 + z_1 \qquad y_1 \in A,\ z_1 \in A^{\perp}$$

$$x_2 = y_2 + z_2 \qquad y_2 \in A \quad z_2 \in A^{\perp}$$

So that

$$\pi(x_1+x_2) = \pi(y_1+y_2+z_1+z_2)$$
$$= y_1+y_2$$
$$= \pi(x_1)+\pi(x_2)$$

Also for any $\alpha \in F$

$$\pi(\alpha x_1) = \pi(\alpha y_1+\alpha z_1)$$
$$= \alpha y_1$$
$$= \alpha \pi(x_1) \qquad \forall x_1 \in H$$

Hence π is linear

Theorem# Let A be a closed subspace of Hilbert space H. If π is the orthogonal Projection of H onto A, then

(1) $\langle \pi(x_1), x_2\rangle = \langle x_1, \pi(x_2)\rangle \quad \forall x_1, x_2 \in H$

(2) $\langle \pi(x), x\rangle = \|\pi(x)\|^2 \leq \|x\|^2$

(3) The null space of π is $A^{\perp}$

(4) If I is identity mapping of H, Then $I-\pi$ is also linear and its null space is A

Proof# Let $x_1, x_2 \in H$. Then there are $y_1, y_2 \in A$ and $z_1, z_2 \in A^{\perp}$ such that

$$x_1 = y_1+z_1 \qquad x_2 = y_2+z_2$$

So $\pi(x_1) = y_1 \qquad \pi(x_2) = y_2$

and

$$\langle \pi(x_1), x_2\rangle = \langle y_1, y_2+z_2\rangle$$
$$= \langle y_1, y_2\rangle + \langle y_1, z_2\rangle$$
$$= \langle y_1, y_2\rangle + 0$$
$$= \langle y_1, y_2\rangle$$

Hence $\langle \pi(x_1), x_2\rangle = \langle x_1, \pi(x_2)\rangle$

(2) # For any $x = y+z \in H \qquad y \in A,\ z \in A^{\perp}$

$$\|x\|^2 = \|y\|^2+\|z\|^2 \qquad \because \langle y, z\rangle = 0$$

So $\langle \pi(x), x\rangle = \langle y, y+z\rangle$

$$= \langle y, y\rangle \longrightarrow ①$$

$$= \|y\|^2$$

$$\leq \|y\|^2 + \|z\|^2$$

$$\leq \|x\|^2 \qquad x = y+z, \quad \|x\|^2 \leq \|y\|^2 + \|z\|^2$$

Also

$$\langle \pi(x), x\rangle = \langle y, y\rangle \quad \text{from ①}$$

$$= \langle \pi(x), \pi(x)\rangle$$

$$= \|\pi(x)\|^2$$

Hence $\langle \pi(x), x\rangle = \|\pi(x)\|^2 \leq \|x\|^2$

(iii) For any $z \in A^\perp$, $\pi(z) = 0$ so that if N_π is the null space of π. Then

$$A^\perp \subseteq N_\pi$$

Conversely let $x \in N_\pi$.

Then $x = y+z \quad y \in A,\ z \in A^\perp$

and

$$0 = \pi(x) = \pi(y+z) = y$$

$$\Rightarrow x = z \in A^\perp$$

Hence $N_\pi \subseteq A^\perp$

Thus

$$A^\perp = N_\pi, \text{ the null space of } \pi$$

(4) Since both I and π are linear, $I-\pi$ is linear. Also for any

$x = y+z$ in H $\quad y \in A$; $z \in A^\perp$, we have.

$$(I-\pi)(x) = I(x) - \pi(x)$$

$$= y+z-y$$

$$= z$$

So that the orthogonal complement of $A^\perp$ is $A^{\perp\perp} = A$ because A is closed. Hence the null space of $I-\pi$ is A, as required.

Invariant Subspace of Hilbert Space

Let T be a linear operator on H i.e $T: H \to H$. A closed subspace A of H is called T-invariant or invariant under T if

$$T(A) \subseteq A$$

when this happens $T|_A$ is linear on A.
If both A & $A^{\perp}$ are invariant under T, we say that A reduces T or that T is reduced by A. This situation is more interesting, for it allows us to replace the study of T as a whole by the study of its restrictions to A & $A^{\perp}$

For example, A is invariant under the projection π

Theorem # Let π be the projection of H onto A, a closed subspace of a Hilbert space H. Suppose that $f: H \to H$ is a linear operator. Then A is invariant under f, iff

$$f\pi = \pi f \pi$$

Proof # Suppose A is invariant under f and $x \in H$. Then $\pi(x) \in A$ so that

$$f\pi(x) \in A$$

Also for any $y \in A$, $\pi(y) = y$

so $(\pi f \pi)(x) = \pi(f\pi)(x) = (f\pi)(x) \quad \forall x \in H$

Hence $\pi f \pi = f\pi$

Conversely suppose that $\pi f \pi = f \pi$ and let $x \in A$. Then from $\pi(x) = x$, we have

$$f(x) = (f\pi)(x) = (\pi f \pi)(x)$$
$$= \pi(f\pi)(x) \text{ is also in } A$$

Hence A is invariant under f, as required.

Problem# Prove that $I - \pi = \pi^*$ is orthogonal to π in the sense that

$$\pi\pi^* = 0 \qquad 0 \text{ is zero function on } H$$

In fact we the following general result

Theorem# Let A and B be closed subspaces of a Hilbert space H and π, π^* projections of H onto A & B respectively. Then

$$A \perp B \quad \text{iff} \quad \pi\pi^* = 0$$

Proof# Suppose that $A \perp B$. Then $B \subseteq A^\perp$
So for any $x \in H$

$$\pi^*(x) \in B \subseteq A^\perp$$

Hence for all $x \in H$

$$(\pi\pi^*)(x) = \pi(\pi^*(x)) = \pi(z) = 0 \quad z = \pi^*(x) \in A^\perp$$

Hence

$$\pi\pi^* = 0$$

Conversely suppose that

$$\pi\pi^* = 0$$

Then for any $x \in B$, $\pi^*(x) = x$ so that

$$\begin{aligned}\pi(x) &= \pi(\pi^*(x)) \\ &= (\pi\pi^*)(x) \\ &= 0\end{aligned}$$

$\Rightarrow x \in A^\perp$. Thus $B \subseteq A^\perp$ i.e

$$A \perp B$$

Linear Operator# Let M, N be normed spaces over the same field F. A function $T: N \to M$ is said to be linear operator if

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) \qquad \forall \alpha, \beta \in F, \ \forall x_1, x_2 \in N$$

Linear functional

A special type of linear operator from a normed space $N(F)$ to F, where F is R or C and is itself a normed space under the usual norm defined on R or C.

Thus $f: N \longrightarrow F$ is said to be linear function if

$$f(ax+by) = af(x) + bf(y) \quad \forall x, y \in N$$
$$\forall \alpha, \beta \in F$$

Remarks # The noun "functional" seems to have originated in the theory of integral equations. It was used to distinguish between a function in the elementry sense defined on a set of numbers and a function (or functional) defined on a set of functions. We always use the word to mean a scalar-valued linear function defined on a normed space. Linear functional is continuous iff it is bounded.

Linear functionals on Hilbert Spaces

Theorem # Let H be a Hilbert space and $y \in H$. Then function

$$f_y : H \longrightarrow F \quad (F \text{ is } R \text{ or } C)$$

given by

$$f_y(x) = \langle x, y \rangle \qquad \forall x \in H$$

defines a linear functional on H. Moreover

$$\|f_y\| = \|y\| \quad , \; y \in H$$

Proof # **Linearity:** Let $\alpha_1, \alpha_2 \in F$ & $x_1, x_2 \in H$

Then

$$f_y(\alpha_1 x_1 + \alpha_2 x_2) = \langle \alpha_1 x_1 + \alpha_2 x_2 \; y \rangle$$
$$= \alpha_1 \langle x_1 \; y \rangle + \alpha_2 \langle x_2 \; y \rangle$$

$= \alpha_1 f_y(x_1) + \alpha_2 f_y(x_2)$

$\Rightarrow f_y$ is linear functional

Norm: we prove that

$\|f_y\| = \|y\|$

By Cauchy schwarz inequality

$$|f_y(x)| = |\langle x, y\rangle| \le \|x\|\,\|y\| \quad \forall x \in H$$

So

$$\|f_y\| \le \|y\| \quad \to (1)$$

Also

$$\|f_y\| = \sup_{\|x\|=1} |f_y(x)|$$

$$\geq |f_y(y/\|y\|)|$$

$$\geq \langle y/\|y\|, y\rangle = \frac{1}{\|y\|}\langle y, y\rangle = \frac{\|y\|^2}{\|y\|}$$

$$\geq \|y\| \quad \to (2)$$

By (1) & (2)

$$\|f_y\| = \|y\|$$

Note # Let $T: N \longrightarrow M$ be a bounded linear operator. Then there is a real no k such that

$$\|Tx\| \le k\|x\| \quad \forall x \in N$$

Suppose that $x \neq 0$. Then

$$\frac{\|Tx\|}{\|x\|} \le k \quad \forall x \in N, x \neq 0$$

$\Rightarrow k$ is an upper bound for $\frac{\|Tx\|}{\|x\|}$. The least upper bound

$\sup\limits_{\substack{x \neq 0 \\ x \in N}} \frac{\|Tx\|}{\|x\|}$ is called the norm of T

i.e

$$\|T\| = \sup_{\substack{x \neq 0 \\ x \in N}} \frac{\|Tx\|}{\|x\|}$$

By defininition of Supremum

$$\frac{\|Tx\|}{\|x\|} \le \|T\|$$

$$\|Tx\| \le \|T\|\,\|x\|$$

Two other relations for a bounded linear operator T are

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\|$$

These follows from

$$\|T\| = \sup_{\substack{y\neq 0 \\ y\in N}} \frac{\|Ty\|}{\|y\|} = \sup_{\substack{y\neq 0 \\ y\in N}} \left\|T\left(\frac{y}{\|y\|}\right)\right\|$$

$$= \sup_{\|x\|=1} \|Tx\| \quad x = \frac{y}{\|y\|}$$

Null Space of Linear functional

Let $f : V \to F$ be a linear functional. Then the set

$$N = \{x \in V : f(x) = 0\}$$

is called null space of f. The null space of f is a closed subspace of V of co dimension 1.

Theorem# (Riesz Representation Theorem)

Let H be a Hilbert space and f be any linear functional in H. Then there is a unique y in H such that

$$f(x) = \langle x\ y\rangle \quad \forall x \in H$$

Proof# If $f = 0$, zero linear functional, then we may take $y = 0$ because in that case

$$0 = f(x) = \langle x\ 0\rangle \quad \forall x \in H$$

So let $f \neq 0$. Then null space of f is proper closed subspace of H

If

$$N^{\perp} = \{z \in H : z \perp N\}$$

Then $N^{\perp} \neq \{0\}$ is a closed subspace of H

and is infact, the orthogonal complement of N.

Let $0 \neq x' \in N^{\perp}$

Then

$$f(x') \neq 0$$

Take
$$y = ax' \quad \rightarrow (1)$$
where a is a scalar determined by

$$f(x') = \langle x' y \rangle$$
$$= \langle x' \, ax' \rangle$$
$$= \bar{a} \, \|x'\|^2$$

$$\bar{a} = \frac{f(x')}{\|x'\|^2}$$

$$a = \frac{\overline{f(x')}}{\|x'\|^2} \quad \rightarrow (2)$$

we verify that y as chosen in (1) with a, as given by (2) satisfies the required condition

Let $x \in H$

put $b = f(x) / f(x')$

Then

$$f(x - bx') = f(x) - b f(x') \quad \because f \text{ is linear}$$
$$= f(x) - \frac{f(x)}{f(x')} \cdot f(x')$$

$$= 0$$

So that $x - bx' \in N$

$\because y \in N^{\perp}$ because $x' \in N^{\perp}$ & $N^{\perp}$ is subspace $\therefore ax' = y \in N^{\perp}$

$\therefore$ we have

$$\langle x, y \rangle = \langle x - bx' \, y \rangle + \langle bx' y \rangle$$
$$= \langle bx' \, y \rangle \quad \because \langle x - bx' \, y \rangle = 0$$

$$= b \langle x' y \rangle$$
$$= b f(x') = \frac{f(x)}{f(x')} \cdot f(x') = f(x)$$

Uniqueness # Suppose that for different y, y' in H

$$f(x) = \langle x, y\rangle \quad \& \quad f(x) = \langle x, y'\rangle$$

$$\Rightarrow \langle x, y\rangle = \langle x, y'\rangle \quad \forall x \in H$$

$$\langle x, y\rangle - \langle x, y'\rangle = 0.$$

$$\langle x, y\rangle + \langle x, -y'\rangle = 0.$$

$$\langle x, y - y'\rangle = 0 \quad \forall x \in H$$

In particular for $x = y - y'$

$$\langle y - y', y - y'\rangle = 0$$

$$\|y - y'\|^2 = 0$$

$$y - y' = 0$$

$$y = y'$$

Hence y is unique.

Conjugate Linear Mapping

A mapping $f: H \to F$ is said to be conjugate linear if

$$f(ax + a'x') = \bar{a} f(x) + \bar{a}' f(x') \quad \forall x, x' \in H$$

and $a, a' \in F$

Corollary # Let H be a Hilbert space and for any y in H, $f_y : H \to F$ be given by

$$f_y(x) = \langle x, y\rangle.$$

Then the correspondence

$$y \xleftrightarrow{\varphi} f_y = y', \quad y'(x) = \langle x, y\rangle$$

is a conjugate linear isometric isomorphism bet H and H', the space of linear functionals on H.

Alternatively

$$H' = \{ f_y : H \to F : f_y(x) = \langle x, y\rangle \text{ for some } y \in H\}$$

Proof # By above theorems every bounded

linear functionals on H, is uniquely of the form f_y defined by

$$f_y(x) = \langle x, y \rangle \quad \forall x \in H$$

for a unique y in H. Hence the correspondence

$$y \overset{\phi}{\longleftrightarrow} f_y \quad \text{is one-one}$$

* Also for y_1, y_2 in H & $a_1, a_2 \in F$

$$\phi(a_1 y_1 + a_2 y_2) = f_{a_1 y_1 + a_2 y_2}$$

where

$$\begin{aligned} f_{a_1 y_1 + a_2 y_2}(x) &= \langle x, a_1 y_1 + a_2 y_2 \rangle \\ &= \bar{a}_1 \langle x, y_1 \rangle + \bar{a}_2 \langle x, y_2 \rangle \\ &= \bar{a}_1 f_{y_1}(x) + \bar{a}_2 f_{y_2}(x) \\ &= (\bar{a}_1 f_{y_1} + \bar{a}_2 f_{y_2})(x) \quad \forall x \in H \end{aligned}$$

Hence

$$\begin{aligned} f_{a_1 y_1 + a_2 y_2} &= \bar{a}_1 f_{y_1} + \bar{a}_2 f_{y_2} \\ &= \bar{a}_1 \phi(y_1) + \bar{a}_2 \phi(y_2) \\ &= \bar{a}_1 \phi(y_1) + \bar{a}_2 \phi(y_2) \end{aligned}$$

So

$$\phi(a_1 y_1 + a_2 y_2) = \bar{a}_1 \phi(y_1) + \bar{a}_2 \phi(y_2)$$

$\Rightarrow \phi$ is conjugate linear

Also for any $y \in H$

$$\|y\| = \|f_y\| = \|y'\|$$

Hence ϕ is an isometric isomorphism between H & H'

Sublinear functional OR Convex functional

A sublinear function p on a vector space X is a real valued function which satisfies the following properties

(i) $p(x+y) \leq p(x) + p(y)$ (Sub additive property)

(2) $p(\alpha x) = |\alpha|\, p(x)$ $\quad \alpha \in R$

(Homogeneous property)

Hahn Banach Theorem

Let X be a vector space and p a sublinear functional on X. Furthermore, let f be a linear function on a subspace Z of X and satisfies the properties

(1) $\quad f(x) \leq p(x) \quad \forall x \in Z$

Then f has a linear extension $\tilde{f}$ from Z to X satisfying

(1)' $\quad \tilde{f}(x) \leq p(x) \quad \forall x \in X$

i.e $\tilde{f}$ is a linear extension on X an X satisfying (1)' and

$$\tilde{f}(x) = f(x) \quad \forall x \in Z$$

OR

Let p be a sublinear functional on X and f a linear functional on $Z \subset X$ such that

$$f(x) \leq p(x) \quad \forall x \in Z$$

Then f has extension $\tilde{f}$ such that

$$\tilde{f}(x) \leq p(x) \quad \forall x \in Z$$

Hahn Banach Generalized Theorem

Let p be a sublinear functional on a real vector space or complex vector space X. Further let f be a linear functional defined on a subspace Z of X satisfying

$$|f(x)| \leq p(x) \quad \forall x \in Z$$

Then f has a linear extension $\tilde{f}$ from Z to X satisfying

$$|\tilde{f}(x)| \leq p(x) \quad x \in X$$